

Approximate Euclidean Shortest Paths amid Polygonal Obstacles

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Abstract

Given a set \mathcal{P} of non-intersecting polygonal obstacles in \mathbb{R}^2 defined with n vertices, we compute a sketch Ω of \mathcal{P} whose size is independent of n . We utilize Ω to devise an algorithm to compute a $(1 + \epsilon)$ -approximate Euclidean shortest path between two points given with the description of \mathcal{P} . When \mathcal{P} comprises of convex polygonal obstacles, we devise a $(2 + \epsilon)$ -approximation algorithm to efficiently answer two-point Euclidean distance queries.

1 Introduction

Given a set $\mathcal{P} = \{P_1, P_2, \dots, P_h\}$ of pairwise-disjoint simple polygonal obstacles in \mathbb{R}^2 and two points s and t in the free space $\mathcal{F}(\mathcal{P})$ defined by the closure of \mathbb{R}^2 sans the union of interior of all the simple polygons in \mathcal{P} , the Euclidean shortest path finding problem asks to compute a shortest path between s and t that lies in $\mathcal{F}(\mathcal{P})$. This problem is well-known in computational geometry community. Mitchell [22] provides an extensive survey of research accomplished in determining shortest paths in polygonal and polytope domains to that date. In the following, we assume that n vertices together define h obstacles of \mathcal{P} .

The polygonal domain \mathcal{P} is given as input, a priori. The three variants of the problem include: (i) both s and t are given as input with \mathcal{P} , (ii) only s is given as input with \mathcal{P} , and (iii) neither s nor t is given as input. The type (i) problem is a single-shot problem and involves no preprocessing. In a type (ii) problem, the preprocessing phase constructs a shortest path map with s as the source so that a shortest path between s and any given query point t can be found efficiently. In the third variation, which is known as a two-point shortest path query problem, \mathcal{P} is preprocessed to construct data structures that facilitate in answering shortest path queries between any given pair of query points s and t .

In solving type (i) or type (ii) problem, there are two fundamentally different approaches: the visibility graph method (see Ghosh [10] for both the survey and details of various visibility algorithms) and the wavefront method. The visibility graph method [23, 18, 17] is based on constructing a graph G whose nodes are the vertices of the obstacles (together with s and t) and edges are the pairs of mutually visible vertices. Once the visibility graph G is available, a shortest path between s and t in G is found using Dijkstra's algorithm. As the number of edges in the visibility graph is $O(n^2)$, this method has a quadratic time complexity barrier. In the wavefront based approach [12, 21, 16, 13], a wavefront is expanded from s till it reaches t . The wavefront method typically constructs a shortest path map with respect to s so that for any query point t , a shortest path from

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s to t can be found efficiently. Considering that the algorithms to find optimal shortest paths are complicated, simple approximation algorithms were devised in [6, 1].

The two-point shortest path query problem within a given simple polygon was addressed by Guibas and Hershberger [11]. Their result preprocesses the given simple polygon in time $O(n)$ and constructs a data structure of size $O(n)$ and answers two-point shortest path distance queries in time $O(\lg n)$. The exact two-point shortest path queries in polygonal domain were explored by Chiang and Mitchell [5]. This result primarily devises two algorithms: one to construct data structures of size $O(n^5)$ that facilitates in answering distance queries in $O(h + \lg n)$ time; the other algorithm builds data structures of size $O(n + h^5)$ and yields $O(h \lg n)$ query time. In both of these algorithms, a shortest path itself is found in additional time $O(k)$, where k is the number of edges in the output path. Guo et al. [7] preprocesses $\mathcal{F}(\mathcal{P})$ in $O(n^2 \lg n)$ time to construct data structure of size $O(n^2)$ to answer two-point Euclidean distance queries for any given pair of query points in $O(h \lg n)$ time.

Because of the difficulty of exact two-point queries in polygonal domains, various approximation algorithms were devised. Clarkson first made such an attempt in [6]. Chen [3] used the techniques from [6] in constructing a data structure of size $O(n \lg n + \frac{n}{\epsilon})$ in $o(n^{3/2}) + O(\frac{n}{\epsilon} \lg n)$ time to support $O(6 + \epsilon)$ -approximate two-point distance queries in $O(\frac{\lg n}{\epsilon} + \frac{1}{\epsilon^2})$ time. Arikati et al. [2] gave a family of results with trade-offs among the approximation factor, preprocessing time, storage space and the query time. Agarwal et al. [1] computes an approximate shortest path in $O(n + \frac{h}{\sqrt{\epsilon}} \lg(\frac{h}{\epsilon}))$ time when the obstacles are convex.

All through this document, to distinguish graph vertices from the vertices of the polygonal domain, we refer to vertices of graph as nodes. The Euclidean distance between any two points p and q is denoted with $\|pq\|$. The obstacle avoiding geodesic distance between any two points p, q amid a set \mathcal{Q} of obstacles is denoted with $\text{dist}_{\mathcal{Q}}(p, q)$. We denote the convex hull of a set R of points or a simple polygon R with $CH(R)$. For any set of polygonal obstacles \mathcal{Q} , the free space resultant from the closure of \mathbb{R}^2 sans the union of interior of all the polygons in \mathcal{Q} is denoted with $\mathcal{F}(\mathcal{Q})$. Let r' and r'' be two rays with origin at p , and respectively make θ' and θ'' counterclockwise angles with the positive x -axis in a coordinate system. Let \vec{v}_1 and \vec{v}_2 be the respective unit vectors corresponding to rays r' and r'' . A *cone* $C_p(r', r'')$ is the set of points defined by rays r' and r'' such that a point $q \in C_p(r', r'')$ if and only if q can be expressed as a convex combination of vectors \vec{v}_1 and \vec{v}_2 . When the rays are obvious from the context, we denote the cone with C_p . The counterclockwise angle from the positive x -axis to the line that bisects the cone angle of C_p is termed as the *orientation of the cone* C_p .

Our contribution

We compute a sketch Ω comprising of simple polygonal obstacles from the set \mathcal{P} of input simple polygonal obstacles. The sketch Ω has h simple polygonal obstacles: for $1 \leq i \leq h$, simple polygon $P_i \in \mathcal{P}$ is approximated with another simple polygon $Q_i \in \Omega$ such that $Q_i \subseteq CH(P_i)$. In computing sketch Ω of \mathcal{P} , we identify coreset S_i of vertices of each polygon $P_i \in \mathcal{P}$. Further, we form a corepolygon $Q_i \in \Omega$, which is again a simple polygon, by joining every successive pairs of vertices of S_i that occur while traversing P_i in counterclockwise order with a line segment. Note that when P_i is convex, the corresponding corepolygon Q_i obtained through this procedure is both convex as well as $Q_i \subseteq P_i$. Like in [1], combinatorial complexity of Ω is independent of the number of vertices defining \mathcal{P} . For two points $s, t \in \mathcal{F}(\mathcal{P})$, we compute an approximate Euclidean path between s and t in $\mathcal{F}(\Omega)$ using an algorithm that is a variant of [6]. From this path, we compute

another path in $\mathcal{F}(\mathcal{P})$ and show that this new path is a $(1 + \epsilon)$ -approximate shortest path between s and t amid polygons in \mathcal{P} . The main contributions of this paper are summarized below:

- * When \mathcal{P} comprises of convex polygonal obstacles, we preprocess those obstacles in $O(n + \frac{h}{\epsilon''} \lg \frac{h}{\sqrt{\epsilon''}} + \frac{h}{\sqrt{\epsilon''}} \lg^2 \frac{h}{\sqrt{\epsilon''}})$ time to construct data structures of size $O(\frac{h}{\sqrt{\epsilon''}})$ to answer two-point $(2 + \epsilon)$ -approximate geodesic Euclidean distance (length) queries in $O(\frac{1}{(\epsilon'')^3} \lg^2 \frac{h}{\sqrt{\epsilon''}})$ time, where $\epsilon'' = (\frac{2+\epsilon}{2})^{1/3} - 1$.
- * As part of devising the above algorithm (i.e., again when \mathcal{P} comprises of convex polygonal obstacles), we devise an algorithm to compute a $(1 + \epsilon)$ -approximate Euclidean distance between two points $s, t \in \mathcal{F}(\mathcal{P})$, given the description of \mathcal{P} , in $O(n + \frac{h}{\sqrt{\epsilon'}} \lg \frac{h}{\sqrt{\epsilon'}})$ time, where $\epsilon' = \sqrt{1 + \epsilon} - 1$. Further, a $(1 + \epsilon)$ -approximate Euclidean shortest path is computed in additional $O(h \lg n)$ time.
- * Further, we extend the above algorithm for computing a $(1 + \epsilon)$ -approximate Euclidean shortest path between s and t in $O(n + h((\lg n) + (\lg h)^{1+\delta} + \frac{1}{\sqrt{\epsilon'}} \lg \frac{h}{\epsilon'}))$ time when \mathcal{P} comprises of simple polygons. Here ϵ' equals to $\sqrt{1 + \epsilon} - 1$, and δ is a small positive constant due to time involved in triangulating $\mathcal{F}(\mathcal{P})$ using [15].

In addition, our algorithm to compute the coreset of simple polygons and obtaining a sketch of \mathcal{P} may be of independent interest in devising efficient approximation algorithms to other geometric optimization problems.

As mentioned above, Agarwal et al. [1] devises a $(1 + \epsilon)$ -approximation algorithm to compute a s - t Euclidean shortest path in $O(n + \frac{h}{\sqrt{\epsilon}} \lg \frac{h}{\epsilon})$ time when obstacles in \mathcal{P} are convex. Our algorithm is applicable even when polygonal obstacles in \mathcal{P} are non-convex. Our approach of computing coresets is quite different from [1]. The key differences between our algorithm and [1] are detailed below. For two-point shortest paths, Chiang and Mitchell [5] outputs an optimal shortest path; our result answers approximate two-point distance queries with time-space trade-offs with respect to [5]. Moreover, our result trades-off with approximation algorithms devised in [2, 3].

In \mathbb{R}^2 , our algorithm is very different from [1]. Let the polygonal domain \mathcal{P} be defined with convex polygons P_1, P_2, \dots, P_h . In this algorithm as well as in [1], P_i is approximated with Q_i , for every $1 \leq i \leq h$. However, for every $1 \leq i \leq h$, in our algorithm $Q_i \subseteq P_i$ whereas in [1], $P_i \subseteq Q_i$.

Let the new polygonal domain Ω be defined with simple polygons Q_1, Q_2, \dots, Q_h . Unlike [1], in computing Ω , our algorithm does not require using plane-sweep algorithm to find pairwise vertically visible simple polygons of \mathcal{P} . Our algorithm partitions the boundary of each convex polygon P into a set of contiguous patches so that the angle subtended by any two points belonging to same patch is upper bounded as a function of ϵ . For any two points $s, t \in \mathcal{F}(\mathcal{P})$, our algorithm guarantees that $\text{dist}_\Omega(s, t) \leq \text{dist}_\mathcal{P}(s, t) \leq (1 + \epsilon) \text{dist}_\Omega(s, t)$. For any two points $s, t \in \mathcal{F}(\Omega)$, algorithm in [1] guarantees that $\text{dist}_\Omega(s, t) \leq (1 + \epsilon) \text{dist}_\mathcal{P}(s, t)$. To find a shortest path amid Ω , our algorithm does not use algorithm from [12]. Instead, our algorithm achieves the said approximation using the spanner constructed from cone Voronoi diagrams (CVDs) [6]. Apart from computing sketch Ω of \mathcal{P} , as compared with [6], number of cones per obstacle that participate in computing CVDs amid $\mathcal{F}(\Omega)$ is further optimized to achieve the above mentioned time and space complexities. This is achieved by exploiting the convexity of obstacles together with the properties of shortest paths amid convex obstacles. Further, in our algorithm, for any maximal line segment with endpoints r', r'' along the computed (approximate) shortest path p amid obstacles in Ω , if $r'r''$ lies in $P_i - Q_i$

for any $1 \leq i \leq h$, then before outputting p we replace line segment $r'r''$ with geodesic shortest path between r' and r'' in $\mathcal{F}(\mathcal{P})$ i.e., with a shortest path between r' and r'' along the boundary of P_i .

More importantly, the definition of coresets and corepolygons of convex polygonal obstacles is extended to simple polygons using the decomposition of $\mathcal{F}(\mathcal{P})$ into hourglasses [14, 18, 16]. And, the sketch Ω is defined to comprise simple polygons, like in the convex polygonal case each $Q_i \in \Omega$ correspond to an obstacle $P_i \in \mathcal{P}$ and is obtained from the coreset S_i of P_i . The scheme designed in Agarwal et al. [1] does not appear to extend easily to the case of simple polygons as they use the critical step of computing partitioning planes between pairs of convex polygonal obstacles from \mathcal{P} .

The algorithm for single-shot approximate shortest path computation when obstacles are convex polygonal is described in Section 2. Section 3 describes algorithm for computing an approximate Euclidean shortest path amid simple polygonal obstacles. The preprocess-query algorithm to answer approximate Euclidean distance queries is described in Section 4. The conclusions are in Section 5.

2 Approximate shortest path amid convex polygons

In this Section, we suppose that every polygon belonging to \mathcal{P} is convex. We use the following notation from Yao [24]. Let \mathcal{C} be the set of cones with disjoint interiors partitioning \mathbb{R}^2 , where each cone has an apex at the origin and the cone angle is upper bounded by a value that is a function of ϵ . Each cone that we refer in this paper is a translated copy of some cone in \mathcal{C} . When a cone $C \in \mathcal{C}$ is translated to have apex at a point p , the translated cone is denoted with C_p . We show that $O(\frac{h}{\sqrt{\alpha\epsilon}})$ vertices (α defined later), selected from the set of vertices defining \mathcal{P} , together with the select set of cones introduced at these vertices, suffice to compute an approximate shortest path between any two points in $\mathcal{F}(\mathcal{P})$ with the desired accuracy.

2.1 Computing a sketch of \mathcal{P}

First, we detail an algorithm to compute a coreset of \mathcal{P} . Let e_j, e_{j+1}, \dots, e_k be a sequence C of edges when the boundary of a convex polygon P_i is traversed in counterclockwise order such that for every $j \leq l \leq k-1$, vertex v_l is common to edges e_l and e_{l+1} . The absolute value of difference in angle made by e_j and e_k with the positive x -axis is defined as the *angle subtended by C* . Let Π_i be a partition of the boundary of a convex polygon P_i into a collection of $\lceil \frac{2\pi}{\sqrt{\alpha\epsilon}} \rceil$ contiguous sections, *patches*, such that the angle subtended by any contiguous section is upper bounded by $\sqrt{\alpha\epsilon}$, with α defined in terms of ϵ later.

Lemma 2.1 *For any two points p and q that belong to any patch $R \in \Pi_j$, the geodesic Euclidean distance between p and q along R is upper bounded by $(1 + \alpha\epsilon)\|pq\|$.*

Proof: Let e' be the edge on which p lies and let e'' be the edge on which q lies. Let c be the point of intersection of normal to e' at p and the normal to e'' at q . Since p and q belong to the same patch, the angle between cp and cq is upper bounded by $\sqrt{\alpha\epsilon}$. The geodesic distance between p and q along R is upper bounded by $\frac{\|pq\|\sqrt{\alpha\epsilon}}{\sin \sqrt{\alpha\epsilon}} \leq \frac{\|pq\|\sqrt{\alpha\epsilon}}{(\sqrt{\alpha\epsilon} - \frac{(\sqrt{\alpha\epsilon})^3}{6})} \leq \frac{\|pq\|}{(1 - \frac{\alpha\epsilon}{6})} \leq (1 + \frac{\alpha\epsilon}{6})\|pq\| \leq (1 + \alpha\epsilon)\|pq\|$. \square

For each obstacle P_i , the *coreset* S_i of P_i comprises of two vertices chosen from each patch in Π_i . In particular, for any patch $\pi \in \Pi_i$ defined with the sequence $v_j, v_{j+1}, \dots, v_{k-1}, v_k$ of vertices along an obstacle P_i , both the vertices v_j and v_k belong to the coreset S_i of P_i . The *coreset* \mathcal{S} of \mathcal{P} is then simply $\bigcup_i S_i$.

Observation 1 *The size of coreset \mathcal{S} of \mathcal{P} is $O(\frac{h}{\sqrt{\alpha\epsilon}})$.*

Noting that the complexity of $CH(S_i)$ is upper bounded by the complexity of P_i , to achieve the efficiency, for every $1 \leq i \leq h$, our algorithm uses *corepolygon* $Q_i = CH(S_i)$ in place of P_i . Let Ω be a set comprising of corepolygons corresponding to each of the polygons in \mathcal{P} . Then Ω is the *sketch* of \mathcal{P} . The following Lemmas show that we can achieve a $(1 + \epsilon)$ -approximation amid \mathcal{P} while using the sketch of \mathcal{P} .

Lemma 2.2 *Let v', v'' be any two vertices of obstacles in Ω . Then, $dist_{\mathcal{P}}(v', v'') \leq (1 + \alpha\epsilon)dist_{\Omega}(v', v'')$.*

Proof: Let v_1 and v_2 be any two successive vertices along a shortest path between v' and v'' amid Ω . Let \mathcal{O} be the set of obstacles intersected by line segment v_1v_2 in \mathcal{P} . Suppose that v_1 and v_2 belong to obstacles P_j and P_k respectively. Since the line segment v_1v_2 does not intersect with the interior of the $CH(S_j)$ or $CH(S_k)$, it intersects with at most one patch belonging to set Π_j of patches of P_j and at most one patch belonging to set Π_k of patches of P_k . (Refer Fig. 1.) Let the line segment v_1v_2 intersect with a patch $R \in \Pi_j$ at a points v_1 and r . Then from Lemma 2.1, the geodesic distance between v_1 and r along R is upper bounded by $(1 + \alpha\epsilon)\|v_1r\|$. Analogously, let the line segment v_1v_2 intersect with a patch $R' \in \Pi_k$ at points v_2 and r' . Then the geodesic distance from v_2 and r' is upper bounded by $(1 + \alpha\epsilon)\|v_2r'\|$. For any convex polygonal obstacle P_l in \mathcal{O} distinct from P_j and P_k , let p', p'' be the points of intersection of v_1v_2 with the boundary of P_l . Since the line segment v_1v_2 does not intersect with the interior of the convex hull of coreset corresponding to P_l , both p' and p'' belong to the same patch, say $R'' \in \Pi_l$. Then again from Lemma 2.1, the geodesic distance from p' and p'' along patch R'' is upper bounded by $(1 + \alpha\epsilon)\|p'p''\|$.

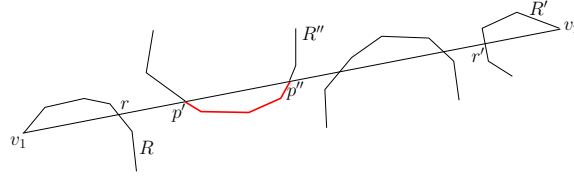


Figure 1: A line segment v_1v_2 of a shortest path amid Ω intersecting three patches belonging to obstacles in \mathcal{P}

Let $\pi_1, \pi_2, \dots, \pi_k$ be the set Π of patches intersected by v_1v_2 , and let p'_i, p''_i be the points of intersections of v_1v_2 with any patch $\pi_i \in \Pi$. Then $\sum_{i=1}^k dist_{\mathcal{P}}(p'_i, p''_i)$ added with $\sum_{i=1}^{k-1} \|p''_i p'_{i+1}\|$ is upper bounded by $(1 + \alpha\epsilon)\|v_1v_2\|$. Let v_1, \dots, v_r be the vertices of \mathcal{P} that occur in that order along a geodesic shortest path between vertices $v', v'' \in \mathcal{P}$ amid Ω . Then $dist_{\mathcal{P}}(v_1, v_r) = \sum_{i=1}^{r-1} dist_{\mathcal{P}}(v_i, v_{i+1}) \leq (1 + \alpha\epsilon) \sum_{i=1}^{r-1} dist_{\Omega}(v_i, v_{i+1})$. \square

Since $\mathcal{F}(\mathcal{P}) \subseteq \mathcal{F}(\Omega)$, every path amid convex polygonal obstacles in \mathcal{P} is also a path amid convex polygonal obstacles in Ω . This observation leads to the following.

Lemma 2.3 *For any two vertices v', v'' of \mathcal{P} , $dist_{\Omega}(v', v'') \leq dist_{\mathcal{P}}(v', v'')$.*

Thus we ensure that for source and destination vertices $s, t \in \mathcal{P}$, shortest path computed amid Ω achieves $(1 + \alpha\epsilon)$ -approximation.

Lemma 2.4 *Let \mathcal{P} be a collection of h convex polygons in \mathbb{R}^2 with n vertices and let s and t be two points in $\mathcal{F}(\mathcal{P})$. Then the sketch \mathcal{S} of \mathcal{P} with cardinality $O(\frac{h}{\sqrt{\alpha\epsilon}})$ suffices to compute a $(1 + \alpha\epsilon)$ -approximate shortest path between s and t in $\mathcal{F}(\mathcal{P})$.*

Proof: Immediate from Observation 1, Lemma 2.2, and Lemma 2.3. \square

2.2 Computing an approximate shortest path using the sketch Ω of \mathcal{P}

Our algorithm relies on [6]; hence, we give a brief overview of the algorithm from [6] to construct a Euclidean spanner $G(V, E)$ from the given set of obstacles \mathcal{P} . Noting that the endpoints of segments of a shortest path in \mathbb{R}^2 comprises of only vertices of \mathcal{P} , the node set V is defined as the vertex set of \mathcal{P} . Let \mathcal{C} be the set of $O(\frac{1}{\epsilon})$ cones apexed at origin that together partition $\mathcal{F}(\mathcal{P})$. Let $C \in \mathcal{C}$ be a cone with orientation θ and let $C' \in \mathcal{C}$ be the cone with orientation $-\theta$. A cone Voronoi diagram CVD is constructed corresponding to each cone $C \in \mathcal{C}$ such that for a cone $C \in \mathcal{C}$ and set K of points, we let $CVD(C, K)$ be a partition of $\mathcal{F}(\mathcal{P})$ where for every point $p \in K$ there is an associated region $R_p \subseteq \mathcal{F}(\mathcal{P})$, $R_p \in CVD(C, K)$. R_p is defined using the cones in \mathcal{C} . Indeed, the region R_p comprises of all points $q \in \mathcal{F}(\mathcal{P})$ such that p is the closest vertex in C'_q among points in K . For a given cone, C_v , let V' be the set of vertices of \mathcal{P} that are visible from v and that lie within the cone C_v . The vertex in V' that is closest to v , say v' , is said to be the closest vertex in C_v to v . For every vertex v of \mathcal{P} and for every cone C_v , an edge e joining v and a closest vertex in C_v to v , say v' , is introduced in E with its weight equal to the Euclidean distance between v and v' . Apart from these, no additional edges are added to E . The result in [6] proves that if d is the obstacle avoiding geodesic Euclidean distance between any two vertices, say v' and v'' , of \mathcal{P} , then the distance between the corresponding nodes v' and v'' in G is upper bounded by $(1+\epsilon)d$. Further, [6] computes $CVD(C, K)$ using plane-sweep in $O(|K| \lg |K|)$ time; and, well-known planar point location structures facilitate locating any point q in any CVD while help in designing shortest path query algorithms.

By limiting the number of vertices of \mathcal{P} at which the cones are initiated to coreset \mathcal{S} of vertices, our algorithm improves the space complexity of the algorithm in [6]. Further, by exploiting the convexity of obstacles, we introduce $O(\frac{1}{\sqrt{\alpha\epsilon}})$ cones per obstacle, each with cone angle $O(\sqrt{\alpha\epsilon})$, and show that these are sufficient to achieve the claimed approximation factor.

Let v_0, v_1, \dots, v_k be vertices such that v_1, \dots, v_{k-1} belong to an obstacle P and v_0, v_k belong to obstacles P' and P'' respectively, for $P \neq P' \neq P''$. Also, let $v_0 v_1 \dots v_k$ be a subpath π of a shortest path. Since P is a convex polygon, $\angle v_j v_{j+1} v_{j+2}$ interior to P is less than π radians for every $j \in [0, k-2]$, i.e. the subpath is *convex w.r.t. P at each of the vertices v_1, v_2, \dots, v_{k-1}* .

Let v be a vertex of \mathcal{P} that belongs to coreset S_i of convex polygon P_i . Let v', v, v'' be the vertices that respectively occur while traversing the boundary of P_i in counterclockwise order. Also, let C' be the cone defined by the pair of rays $(\overrightarrow{vv'}, -\overrightarrow{vv''})$ and let C'' be the cone defined by the pair of rays $(\overrightarrow{vv'}, -\overrightarrow{vv'})$. For a coreset vertex $v \in \mathcal{S}$, a cone $C \in \mathcal{C}$ is said to be *admissible* at v whenever $C_v \cap C'$ or $C_v \cap C''$ is non-empty. (See Fig. 2.) Note that any shortest path is convex at v with respect to P_i . Thus if q is any point in $\mathcal{F}(\mathcal{P})$ such that q is not visible to p amid \mathcal{P} and a shortest path between p and q passes through vertex v of P_i , then there exists a shortest path from p to q such that one of its line segment lies in C' and another line segment of that path lies in C'' . Hence, in computing geodesic shortest path amid \mathcal{P} , it suffices to consider admissible cones at vertices of \mathcal{P} .

Note that whenever two points s and t between which we intend to find a shortest path are

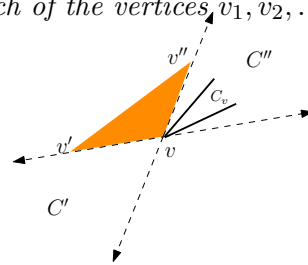


Figure 2: Illustrating an admissible cone C_v incident to a coreset vertex v of an obstacle

visible to each other, the line segment st needs to be computed. To facilitate this, for every degenerate point obstacle p , every cone C with apex p is considered to be an admissible cone.

The same properties carry over to the domain Ω as well. For any two points p_1 and p_2 in $\mathcal{F}(\Omega)$, suppose p_1 and p_2 are not visible to each other. Consider any shortest path π between p_1 and p_2 . For any line segment ab in π , ab is either an edge of Ω or it is a tangent to an obstacle O . In the latter case, ab belongs to an admissible cone of O .

The following Lemma upper bounds the number of cones introduced.

Lemma 2.5 *The number of cones introduced at all the obstacles of Ω together is $O(\frac{h}{\sqrt{\alpha\epsilon}})$.*

Proof: Let \vec{r} be a ray with origin of the coordinate axes as its endpoint. (See Fig. 3.) For any two distinct vertices v' and v'' of a convex polygon P , let $\vec{r}_{v'}$ be the ray parallel to \vec{r} with origin at v' and pointing in the same direction as \vec{r} and let $\vec{r}_{v''}$ be the ray parallel to r with origin at v'' and point in the same direction as r . Also, let v'_1 precede v' (resp. v''_1 precede v'') and v'_2 succeed v' (resp. v''_2 succeed v'') while traversing P in counterclockwise order. Since P is a convex polygon, if $\vec{r}_{v'}$ belong to cone defined by $\vec{v'_1v'}$ and $\vec{v'v'_2}$ then $\vec{r}_{v''}$ is guaranteed to not belong to cone defined by $\vec{v''_2v''}$ and $\vec{v''v''_1}$.

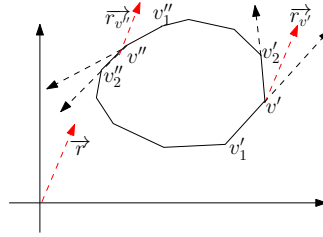


Figure 3: Illustrating that a ray parallel to r can exist in only one admissible cone per obstacle

Extending this argument, if a cone $C_{v'}$ is admissible at v' then the cone $C_{v''}$ cannot be admissible at v'' . Since the number of coreset vertices per obstacle is $O(\frac{1}{\sqrt{\alpha\epsilon}})$, the number of cones introduced per obstacle is $O(\frac{1}{\sqrt{\alpha\epsilon}})$. Further, since there are h convex polygonal obstacles, number of cones at all the obstacle vertices together is $O(\frac{h}{\sqrt{\alpha\epsilon}})$. \square

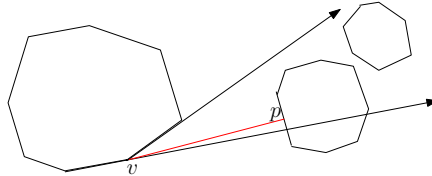


Figure 4: Illustrating an edge of the spanner

Next, we describe computing the spanner $G(V = S \cup S', E)$. The set S comprises of nodes corresponding to coreset \mathcal{S} . For every $v \in S$ and for every admissible cone C_v , let V' be the set of points on the boundaries of obstacles of Ω that are visible from v and lie in C_v . (See Fig. 4.) The point in V' that is closest to v , say p , the *closest Steiner point in C_v to v* is found and p is added to S' . An edge e between v and p is introduced in E while the Euclidean distance from v and v' is set as the weight of e in G . Let p be located on a convex polygonal obstacle P . Further, for every Steiner point p , let v' (resp. v'') be the coreset vertex or Steiner point that lie on the boundary of P and occurs before (resp. after) p while traversing the boundary of P in counterclockwise order. Then an edge e' (resp. e'') between p and v' (resp. p and v'') is introduced in E while the Euclidean

distance from p to v' (resp. p and v'') is set as the weight of e' (resp. e'') in G . The set S' comprises of all such Steiner points. Note that both $|V|$ and $|E|$ are $O(\frac{h}{\sqrt{\alpha\epsilon}})$.

Lemma 2.6 *Let G be the spanner constructed from Ω . Let $\text{dist}_G(p', p'')$ be the distance between p' and p'' in G . Then for any two points $s, t \in \mathcal{F}(\Omega)$, $\text{dist}_\Omega(s, t) \leq \text{dist}_G(s, t) \leq (1 + \alpha\epsilon)\text{dist}_\Omega(s, t)$.*

Proof: Theorem 2.5 of [6] concludes that to achieve $(1 + \alpha\epsilon)$ -approximation, $\sin \psi - \cos \psi \leq \frac{-1}{1 + \alpha\epsilon}$. Expanding *sine* and *cosine* functions for the first few terms yield $-1 + \psi + \frac{\psi^2}{2!} \leq \frac{-1}{1 + \alpha\epsilon}$. Solving the quadratic equation in ψ yields $\psi \leq \sqrt{\alpha\epsilon}$. Hence, the choice of the cone angle of cones and the cardinality of \mathcal{C} in Lemma 2.5.

We claim that introducing a subset of cones (admissible cones) rather than all the cones as used in [6] does not affect the correctness. Let p and q be vertices of convex polygons C_p and C_q respectively. Suppose pq is a line segment belonging to a shortest path in the spanner computed in [6]. If pq is a tangent to convex polygonal obstacle P_i at p , then pq belong to an admissible cone C_p at p . Similarly, if pq is a tangent to convex polygonal obstacle P_j at q , then pq belong to an admissible cone at q . Otherwise, there exists a line segment in admissible cone apexed either at a vertex of P_i or at a vertex of P_j which yield a shorter path from source s to q without using pq line segment. \square

Once we find a shortest path SP_Ω amid obstacles in Ω using spanner G , as for the proof of Lemma 2.3, we transform it to a shortest path amid obstacles in \mathcal{P} . Since there are $O(h)$ obstacles in Ω , SP_Ω contains $O(h)$ tangents. Let this set of tangents be \mathcal{T} . We need to find points of intersection of convex polygons in \mathcal{P} with line segments in \mathcal{T} . Whenever a line segment $l \in \mathcal{T}$ and a convex polygon $P_i \in \mathcal{P}$ intersect, say at points p' and p'' , we replace the line segment between p' and p'' with the geodesic shortest path between p' and p'' along the boundary of P_i . Analogously, for every line segment $l \in SP_\Omega - \mathcal{T}$ belonging to an obstacle $P_j \in \Omega$, we replace l with the corresponding geodesic path along the boundary of P_j . We use the plane-sweep technique [8] to find the points of intersections of line segments in \mathcal{T} with the convex obstacles in \mathcal{P} .

As part of plane-sweep, a vertical line is swept from left-to-right in the plane. Let L (resp. R) be the set of leftmost (resp. rightmost) vertices of convex polygons in \mathcal{P} . Initially, points in L and R together with the two endpoints of every line segment in \mathcal{T} are inserted into the priority queue, say Q . The event points are scheduled from Q using their distance from where the sweep line is initially placed as priority. As the algorithm progress, the event points corresponding to L, R , and the endpoints of line segments in \mathcal{T} are deleted from Q . The algorithm terminates whenever the Q is empty. As described below, the intersection points between the line segments in \mathcal{T} and the convex polygons in \mathcal{P} are added to Q with the traversal of sweep line. The sweep line status is maintained as a balanced binary search tree B . We insert (resp. delete) a line segment in \mathcal{T} or convex polygon in \mathcal{P} , say s , to B whenever leftmost (resp. rightmost) endpoint of s is popped from Q . Since before an intersection occurs between a line segment l from \mathcal{T} and a convex polygon P from \mathcal{P} , it is guaranteed that l and P occur adjacent along the sweep line, we update event-point schedule with an intersection between l and P whenever l and P are adjacent in the sweep line status. By using the algorithm from Dobkin et al. [9], we compute the possible intersection between l and P . If they do intersect, we push the leftmost point of their intersection to Q with the distance from initial sweep line as the priority of that event point. Further, we store the rightmost intersection point between l and P with the leftmost point of intersection as satellite data. If the leftmost intersection point between l and P pops from Q , we compute the geodesic shortest path along the boundary of P between the leftmost intersection point and the corresponding rightmost

intersection point. Further, whenever l and P become non-adjacent along the sweep line, we delete their leftmost point of intersection from Q .

Theorem 2.1 *Given \mathcal{P} and two points s and t in $\mathcal{F}(\mathcal{P})$, computing a $(1 + \epsilon)$ -approximate distance between s and t takes $O(n + \frac{h}{\sqrt{\epsilon'}} \lg \frac{h}{\epsilon'})$ time for $\epsilon' = \sqrt{1 + \epsilon} - 1$. Further, within an additional $O(h \lg n)$ time, a $(1 + \epsilon)$ -approximate shortest path is computed.*

Proof: From Lemma 2.4, we know that $\text{dist}_{\mathcal{P}}(s, t) \leq \text{dist}_{\Omega}(s, t) \leq (1 + \alpha\epsilon)\text{dist}_{\mathcal{P}}(s, t)$. Let G be the spanner constructed. From Lemma 2.6, we know that $\text{dist}_{\Omega}(s, t) \leq \text{dist}_G(s, t) \leq (1 + \alpha\epsilon)\text{dist}_{\Omega}(s, t)$. Combining these two inequalities yields $\text{dist}_{\mathcal{P}}(s, t) \leq \text{dist}_G(s, t) \leq (1 + \alpha\epsilon)^2 \text{dist}_{\mathcal{P}}(s, t)$. To achieve $(1 + \epsilon)$ -approximation, we set $\alpha = \frac{1}{\epsilon}(\sqrt{1 + \epsilon} - 1)$. From here on, we denote $\alpha\epsilon$ with ϵ' .

Finding coresets and computing corepolygons takes $O(n)$ time. The number of coreset vertices is $O(\frac{h}{\sqrt{\epsilon'}})$. The number of cones per obstacle is $O(\frac{1}{\sqrt{\epsilon'}})$. Therefore, the total number of cones is $O(\frac{h}{\sqrt{\epsilon'}})$. For any cone $C \in \mathcal{C}$ and for any corepolygon $O \in \Omega$, at most a constant number of vertices of O are apices to cones that have the orientation of C . Considering a sweep-line in the orientation of C , the sweep-line algorithm to find the closest Steiner point to apex of each cone C (whenever an obstacle intersects with C) takes $O(h \lg h)$ time. Hence, computing closest Steiner points corresponding to all the cone orientations in \mathcal{C} together take $O(\frac{h}{\sqrt{\epsilon'}} \lg h)$.

The number of nodes in the spanner G is $O(\frac{h}{\sqrt{\epsilon'}})$. This include coresets and at most one closest Steiner point per cone. As each cone introduces at most one edge into G , the number of edges in G is $O(\frac{h}{\sqrt{\epsilon'}})$. Finding a shortest path between s and t in G takes $O(\frac{h}{\sqrt{\epsilon'}} \lg \frac{h}{\sqrt{\epsilon'}})$ time. Hence, computing the $(1 + \epsilon')$ -approximate distance between s and t takes $O(n + \frac{h}{\sqrt{\epsilon'}} \lg \frac{h}{\sqrt{\epsilon'}})$ time.

For the plane sweep, leftmost and rightmost extreme vertices of convex polygons in \mathcal{P} are found in $O(n)$ time. There are $O(h)$ line segments in \mathcal{T} , cardinality of Ω is $O(h)$, and $O(h)$ line segment-obstacle pairs (respectively from \mathcal{T} and \mathcal{P}) that intersect. The number of event points due to L, R , and endpoints of line segments in \mathcal{T} is $O(h)$. If l and P become non-adjacent along the sweep-line, deleting their point of intersection from Q is charged to the event that caused them non-adjacent. The sweep-line status structure is updated at the points in R as well as with the rightmost endpoints of line segments in \mathcal{T} ; there are $O(h)$ such event points. Analogous to the analysis provided for line segment intersection [8], our plane sweep algorithm takes $O(n + h \lg h)$ time.

Due to Dobkin et al. [9], determining whether a line segment l in SP_{Ω} intersects with an obstacle P takes $O(\lg n)$ time. The preprocessing structures corresponding to [9] take $O(n)$ space and they are constructed in $O(n)$ time. Further, replacing every line segment between points of intersection with the corresponding geodesic shortest path along the boundaries of obstacles together takes $O(n)$ time altogether. \square

3 Computing approximate shortest path in polygonal domain

In this section, we extend the approximation method to the case of non-convex polygons. We reduce the problem of computing approximate Euclidean shortest path amid simple polygonal obstacles to that of computing the approximate Euclidean shortest path amid convex polygonal obstacles. This is accomplished by first decomposing $\mathcal{F}(\mathcal{P})$ into a set of corridors, funnels, hourglasses, and junctions [14, 18, 16]. First, we describe these geometric structures and then detail how these help in the reduction.

For convenience, obstacles of \mathcal{P} are assumed to be contained in a rectangle \mathcal{R} . Let $Tri(\mathcal{F})$ denote a triangulation of $\mathcal{F}(\mathcal{P})$. (Refer Fig. 5.) The line segments of $Tri(\mathcal{F})$ that are not obstacle edges are referred to as *diagonals*. Let $G(\mathcal{F})$ denote the dual graph of $Tri(\mathcal{F})$, i.e., each node of $G(\mathcal{F})$ corresponds to a triangle of $Tri(\mathcal{F})$ and each edge connects two nodes corresponding to two triangles sharing a diagonal of $Tri(\mathcal{F})$. Based on $G(\mathcal{F})$, we compute a planar 3-regular graph, denoted by G^3 (the degree of every node in G^3 is three), possibly with loops and multi-edges, as follows. First, we remove each degree-one node from $G(\mathcal{F})$ along with its incident edge; repeat this process until no degree-one node remains in the graph. Second, remove every degree-two node from $G(\mathcal{F})$ and replace its two incident edges by a single edge; repeat this process until no degree-two node remains. The resulting graph is G^3 (refer Fig. 5), which has $O(h)$ faces, nodes, and edges. Every node of G^3 corresponds to a triangle in $Tri(\mathcal{F})$, called a *junction triangle*. (Refer Fig. 5.) The removal of the nodes for all junction triangles from G^3 results in $O(h)$ *corridors*, each of which corresponds to an edge of G^3 . This procedure leaves points s and t between which geodesic Euclidean shortest path needs to be computed are to be in their own corridors.

The boundary of each corridor C consists of four parts (see Fig. 6): (1) A boundary portion of an obstacle $P_i \in \mathcal{P}$, from a point a to a point b ; (2) a diagonal of a junction triangle from b to a point e on an obstacle $P_j \in \mathcal{P}$ ($P_i = P_j$ is possible); (3) a boundary portion of the obstacle P_j from e to a point f ; (4) a diagonal of a junction triangle from f to a . The corridor C is a simple polygon. Let $\pi(a, b)$ (resp., $\pi(e, f)$) be the Euclidean shortest path from a to b (resp., e to f) in C . The region H_C bounded by $\pi(a, b)$, $\pi(e, f)$, \overline{be} , and \overline{fa} is called an *hourglass*, which is *open* if $\pi(a, b) \cap \pi(e, f) = \emptyset$ and *closed* otherwise. (Refer Fig. 6.) If H_C is open, then both $\pi(a, b)$ and $\pi(e, f)$ are convex polygonal chains and are called the *sides* of H_C ; otherwise, H_C consists of two *funnels* and a path $\pi_C = \pi(a, b) \cap \pi(e, f)$ joining the two apices of the two funnels, and π_C is called the *corridor path* of C . The paths $\pi(b, x)$, $\pi(e, x)$, $\pi(a, y)$, and $\pi(f, y)$ are termed *sides of funnels* of hourglass H_C . The sides of funnels are convex polygonal chains. For any obstacle $P_j \in \mathcal{P}$, let $\pi(b, x)$ be a side S of a funnel F of H_C such that b is a vertex of P_j . Let $\pi(b, x')$ be a maximal subpath of $\pi(b, x)$ such that x' is a vertex of P_j . Then x' is termed a *pseudo-apex* of side S of funnel F . Note that pseudo-apex of a side S could be same as the apex of the funnel F . Let x', x'' be two pseudo-apices of a closed hourglass H_C such that x' and x'' are vertices of P_j . The shortest path between x' and x'' along the boundary of H_C is the *corridor path between pseudo-apices*.

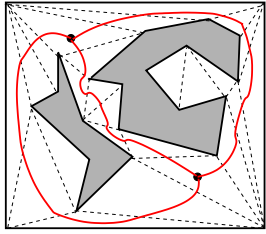


Figure 5: Illustrating a triangulation of the free space among two obstacles and the corridors (indicated by red solid curves). There are two junction triangles marked by a large dot inside each of them, connected by three solid (red) curves. Removing the two junction triangles results in three corridors.

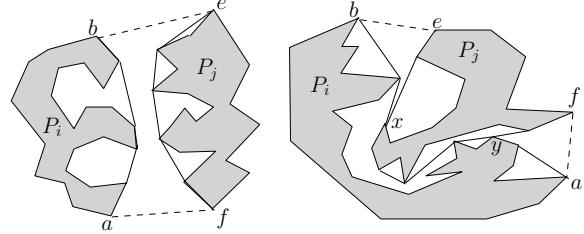


Figure 6: Illustrating an open hourglass (left) and a closed hourglass (right) with a corridor path connecting the apices x and y of the two funnels. The dashed segments are diagonals.

For any obstacle $P_j \in \mathcal{P}$, let \mathcal{C}_j be the union of the following: set of open hourglass sides whose endpoints are incident to P_j , set of maximal sections of sides of funnels whose endpoints are incident to P_j , and the set of line segments each of which joins pseudo-apices that correspond to H_C . The

closed and bounded polygon defined by polygonal chains in \mathcal{C}_j contains P_j ; hence, we term this new polygon as the *expanded polygon of P_j* . If every chain in \mathcal{C}_j is convex, then $\bigcup_{C \in \mathcal{C}_j} C$ is $CH(P_j)$. Since s and t lie exterior to closed hourglasses, as described below, in forming the spanner graph G , we handle corridor paths between pseudo-apices as a special case and introduce corresponding edges into G .

For every convex polygonal chain C of every open hourglass that correspond to any polygonal obstacle P_j , endpoints of C are chosen to be in the coreset S_j corresponding to P_j . Further, similar to the case of convex polygonal obstacles, we compute a coreset of every such convex polygonal chain C and replace C with the convex polygonal chain C' that passes through those coreset of vertices. Essentially, we partition C into patches with angle subtended by each patch to be same as in the case of convex polygonal obstacles. (For details, refer Section 2.) For every closed hourglass H_C that has a vertex common to P_j , both the pseudo-apices of H_C that incident to P_j are included into coreset of P_j . Similar to convex polygonal chains of open hourglasses, coresets corresponding to maximal sections of sides of funnels whose endpoints incident to P_j are computed; and, convex polygonal chain that passes through these coreset of vertices of C is used. Excluding line segments joining successive pseudo-apices, the convex polygonal chains that pass through coreset vertices of each polygonal chain of P_j bound a closed region termed *corepolygon P'_j of P_j* . Since there are $O(h)$ open and closed corridors together and since there are $O(1)$ convex sides per corridor the following is immediate.

Observation 2 *The size of coreset \mathcal{S} of \mathcal{P} is $O(\frac{h}{\sqrt{\alpha\epsilon}})$.*

Similar to convex polygonal obstacle case, we construct a spanner $G(V, E)$ that correspond to corepolygons of \mathcal{P} using CVDs. It is immediate to note that Clarkson's method extends to corepolygons defined as above. And, for each line segment that joins two pseudo-apices along a corepolygon, an edge $e(v', v'')$ is introduced into G : here, v' and v'' are the nodes corresponding to endpoints of C ; and, the weight of e is set as the distance between v' and v'' along C , which is the length of C . Since both the pseudo-apices are included in the coreset of the corresponding obstacle, $v', v'' \in V$. For a shortest path p between any two nodes of G , before outputting p , for every edge $e \in p$ if both the endpoints of e correspond to pseudo-apices a', a'' of a closed hourglass then we replace p with the shortest path along the corridor path between a' and a'' . Computing hourglasses of $\mathcal{F}(\mathcal{P})$ using [14, 18, 16] takes $O(n + h(\lg h)^{1+\delta} + h \lg n)$ time (where δ is a small positive constant due to time involved in triangulating $\mathcal{F}(\mathcal{P})$ using [15]). Extending the proof of Theorem 2.1 leads to the following.

Theorem 3.1 *Given a set \mathcal{P} of polygonal obstacles and two points s and t in $\mathcal{F}(\mathcal{P})$, computing a $(1+\epsilon)$ -approximate Euclidean shortest path between s and t in $O(n + h((\lg n) + (\lg h)^{1+\delta} + (\frac{1}{\sqrt{\epsilon'}} \lg \frac{h}{\epsilon'})))$ time. Here ϵ' is $\sqrt{1+\epsilon} - 1$, and δ is a small positive constant.*

4 Two-point approximate distance queries amid convex polygons

Our query algorithm constructs an auxiliary graph from the spanner network computed during the preprocessing phase of the algorithm. Like in the previous section, our preprocessing algorithm relies on [6]. We compute the approximate distance between the two query points using a shortest path finding algorithm in the auxiliary graph.

4.1 Preprocessing

The graph G constructed as part of preprocessing in Section 2.2 is useful in finding a geodesic shortest path between any two vertices in \mathcal{P} . However, since the time involved in finding a shortest path between nodes in G , or the space to save all-pair-shortest-paths in G are resource intensive, we compute a planar graph $G^{pl}(V, E^{pl})$ from $G(V, E)$ using the result from Chew [4]. Chew's algorithm finds a set $E^{pl} \subseteq E$ in $O(|V| \lg |V|)$ time so that the distance between any two nodes of G^{pl} is a 2-approximation of the distance between the corresponding nodes in G . We use the algorithm from Kawarabayashi et al. [19] to efficiently answer $(1 + \epsilon)$ -approximate distance (length) queries in G^{pl} . More specifically, [19] takes $O(|V| \lg^2 |V|)$ time to construct a data structure of size $O(|V|)$ so that $(2 + \epsilon)$ -approximate shortest distance queries are answered in $O(\frac{\lg^2 |V|}{\epsilon^2})$ time.

Lemma 4.1 *Let G be the spanner computed for the polygonal domain Ω using the algorithm mentioned in Subsection 2.2. Let s, t be two points in $\mathcal{F}(\mathcal{P})$. Let G^{pl} be the planar graph constructed from G using [4]. Further, let $\text{dist}_K(s, t)$ be the distance between s and t in G^{pl} computed using the algorithm from [19]. With appropriately chosen parameters, $\text{dist}_{\mathcal{P}}(s, t) \leq \text{dist}_K(s, t) \leq (2 + \epsilon)\text{dist}_{\mathcal{P}}(s, t)$.*

Proof: From Lemma 2.4, we know that $\text{dist}_{\mathcal{P}}(s, t) \leq \text{dist}_{\Omega}(s, t) \leq (1 + \alpha\epsilon)\text{dist}_{\mathcal{P}}(s, t)$. Let $\text{dist}_G(s, t)$ be the distance in G between nodes s and t of G . From Lemma 2.6, we know that $\text{dist}_{\Omega}(s, t) \leq \text{dist}_G(s, t) \leq (1 + \alpha\epsilon)\text{dist}_{\Omega}(s, t)$. Let $\text{dist}_{G^{pl}}(s, t)$ be the distance in G^{pl} between nodes s and t of G^{pl} . From [4], $\text{dist}_G(s, t) \leq \text{dist}_{G^{pl}}(s, t) \leq 2\text{dist}_G(s, t)$. Then, as mentioned above, $\text{dist}_{G^{pl}}(s, t) \leq \text{dist}_K(s, t) \leq (1 + \alpha\epsilon)\text{dist}_{G^{pl}}(s, t)$.

Combining these, $\text{dist}_{\mathcal{P}}(s, t) \leq \text{dist}_K(s, t) \leq (1 + \alpha\epsilon)^3(2)\text{dist}_{\mathcal{P}}(s, t)$. To achieve $(2 + \epsilon)$ -approximation, we set α to $\frac{1}{\epsilon}((\frac{2 + \epsilon}{2})^{1/3} - 1)$. Since $\alpha < -1$ for $\epsilon < 1$, algorithm yields a $(2 + \epsilon)$ -approximation. \square

In the following, we denote $\alpha\epsilon = (\frac{2 + \epsilon}{2})^{1/3} - 1$ with ϵ'' . From here on, we suppose that there are $O(\frac{1}{\sqrt{\epsilon''}})$ cones in \mathcal{C} , each cone with a cone angle $O(\sqrt{\epsilon''})$. It remains to describe data structures that need to be constructed during the preprocessing phase so as to obtain the closest vertex of query point s (resp. t) in a given cone C_s (resp. C_t). The significance of the same is explained later. To efficiently determine all these $O(\frac{1}{\sqrt{\epsilon''}})$ neighbors to s and t during query time, we construct a set of $O(\frac{1}{\sqrt{\epsilon''}})$ cone Voronoi diagrams (CVDs), each of which correspond to a cone in \mathcal{C} . For any cone $C \in \mathcal{C}$ with orientation ψ , the CVD in that orientation is denoted with CVD_{ψ} . For each $C \in \mathcal{C}$, we consider a sweep-line orthogonal to the orientation of C . The sweep-line algorithm details are same as mentioned in Subsection 2.2.

Lemma 4.2 *The preprocessing phase takes $O(n + \frac{h}{\epsilon''} \lg \frac{h}{\sqrt{\epsilon''}} + \frac{h}{\sqrt{\epsilon''}} \lg^2 \frac{h}{\sqrt{\epsilon''}})$ time. The space complexity of the data structures constructed by the end of preprocessing phase is $O(\frac{h}{\sqrt{\epsilon''}})$. Here, ϵ'' is $(\frac{2 + \epsilon}{2})^{1/3} - 1$.*

Proof: Computing sketch Ω of \mathcal{P} takes $O(n + \frac{h}{\sqrt{\epsilon''}})$ time. The number of cones in all CVDs is $O(\frac{h}{\sqrt{\epsilon''}})$. Constructing spanner G involve computing CVDs, together it takes $O(\frac{1}{\sqrt{\epsilon''}} \frac{h}{\sqrt{\epsilon''}} \lg \frac{h}{\sqrt{\epsilon''}})$ time to compute G . Due to [4], computing planar graph G^{pl} with $O(\frac{h}{\sqrt{\epsilon''}})$ nodes takes $O(\frac{h}{\sqrt{\epsilon''}} \lg \frac{h}{\sqrt{\epsilon''}})$ time. Computing space-efficient data structures using [19] take $O(\frac{h}{\sqrt{\epsilon''}} \lg^2 \frac{h}{\epsilon''})$ time. Hence, the preprocessing phase takes $O(n + \frac{h}{\epsilon''} \lg \frac{h}{\sqrt{\epsilon''}} + \frac{h}{\sqrt{\epsilon''}} \lg^2 \frac{h}{\sqrt{\epsilon''}})$ time.

Further, data structures constructed using [19] by the end of preprocessing phase occupy $O(\frac{h}{\sqrt{\epsilon''}})$ space. Using Kirkpatrick's point location [20], data structures for planar point location are of

$O(\frac{h}{\sqrt{\epsilon''}})$ space complexity. \square

4.2 Shortest distance query processing

The query algorithm finds the geodesic distance between any two given points $s, t \in \mathcal{F}(\mathcal{P})$. We construct a graph G_{st} from G^{pl} by introducing query points s and t as nodes in G_{st} . For every $C \in \mathcal{C}$, whenever C_s (the cone with apex s) intersects an obstacle, we determine a point p on an obstacle, say P_i , that is closest to s in C_s . We add an edge between s and p in G_{st} with weight equal to the Euclidean distance between s and p . Let p lie between two successive coreset vertices, say p_1 and p_2 of P_i . We add edges between p and p_1 and between p and p_2 with the respective geodesic distances along the boundary of obstacle P_i . The set V_s comprises of all such neighbors of s (like p) whereas each such p correspond to a cone $C \in \mathcal{C}$. Analogously, we define neighbors to t in G_{st} .

The node set of G_{st} is $V_s \cup V_t \cup \{s, t\}$. The edges of this graph are of three kinds: $\{s\} \times V_s$, $V_s \times V_t$ and $\{t\} \times V_t$. For every edge (s, s') with $s' \in V_s$, the weight of edge (s, s') is as defined above. The same is true with the edges of type (t, t') with $t' \in V_t$. Note that every node $v \in V_s \cup V_t$ correspond to some vertex in coreset \mathcal{S} . For every edge (s', t') with $s' \in V_s$ and $t' \in V_t$, the weight of (s', t') is the $(2+\epsilon)$ -approximate distance between s' and t' . These weights are obtained from data structures maintained due to [19].

We apply Fredman-Tarjan algorithm to find a shortest path between s and t in G_{st} . It follows from the above discussion, that this distance is a $(2+\epsilon)$ -approximate distance from s to t amid \mathcal{P} . Computing shortest path between s and t in this way effectively equivalent to having both s and t available before preprocessing \mathcal{P} . Therefore, the following is immediate from Lemma 4.1.

Lemma 4.3 *The shortest path obtained between s and t in G_{st} is a $(2+\epsilon)$ -approximate shortest path between s and t amid set \mathcal{P} of convex polygons.*

Lemma 4.4 *The time to query the $(2+\epsilon)$ -approximate distance between two vertices in \mathcal{P} is $O(\frac{1}{(\epsilon'')^3} \lg^2 \frac{h}{\sqrt{\epsilon''}})$.*

Proof: Locating a query point s (resp. t) in any one CVD_ψ takes $O(\lg \frac{h}{\sqrt{\epsilon''}})$ time. Since there are $O(\frac{1}{\sqrt{\epsilon''}})$ CVDs and since finding a closest Steiner point (or, vertex) in each such CVD takes $O(\lg \frac{h}{\sqrt{\epsilon''}})$ time, computing all the closest Steiner points (or, vertices) of s and t together take $O(\frac{1}{\sqrt{\epsilon''}} \lg \frac{h}{\sqrt{\epsilon''}})$. The number of nodes and the number of edges of G_{st} are respectively $O(\frac{1}{\sqrt{\epsilon''}})$ and $O(\frac{1}{\epsilon''})$. Using [19], finding the distance between any two nodes in G^{pl} take $O(\frac{1}{(\epsilon'')^2} \lg^2 \frac{h}{\sqrt{\epsilon''}})$ time. There are $O(\frac{1}{\epsilon''})$ pairs of neighbors of s and t that needs to be considered in computing approximate distances in G^{pl} . Fredman-Tarjan algorithm to compute a shortest path in G_{st} from s to t takes $O(\frac{1}{\epsilon''} \lg^* \frac{1}{\sqrt{\epsilon''}})$. The Theorem follows when these asymptotic time complexities are added up. \square

Theorem 4.1 *Given a set \mathcal{P} of h pairwise-disjoint convex polygons of total complexity n in \mathbb{R}^2 , preprocess \mathcal{P} in $O(n + \frac{h}{\epsilon''} \lg \frac{h}{\sqrt{\epsilon''}} + \frac{h}{\sqrt{\epsilon''}} \lg^2 \frac{h}{\sqrt{\epsilon''}})$ time to build a data structure of size $O(\frac{h}{\sqrt{\epsilon''}})$. For any two query points $s, t \in \mathcal{F}(\mathcal{P})$, a $(2+\epsilon)$ -approximate distance is found in $O(\frac{1}{(\epsilon'')^3} \lg^2 \frac{h}{\sqrt{\epsilon''}})$ time. Here, $\epsilon'' = (\frac{1+\epsilon}{2})^{1/3} - 1$.*

5 Conclusions

We have presented an algorithm to compute a $(1 + \epsilon)$ -approximate Euclidean shortest path amid polygonal obstacles in \mathbb{R}^2 . The time and space complexities of our algorithm better the algorithms devised in Chen [3]. We also devised a $(2 + \epsilon)$ -approximation algorithm to answer two-point Euclidean distance queries. Our result trades-off with approximation algorithms devised in [2, 3]. These algorithms rely on computing a sketch of the polygonal domain by extracting coresets of polygonal obstacles and further computing corepolygons from each of these coresets.

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